Intros and Announcements

Introductions
• Everybody’s name
• Thoughts on Pset 3? Questions?

Announcements
• Next two weeks, I will be gone, but you get a sampling of the other TAs!
  • Adele will be here September 26
  • Max will be here October 3
Plan for Today

• Issues from Pset 2
• Lecture 3 Review
• Discuss Readings
• Problem Set 3
Problem Set 2 Issues

- Problem 3 – Creating a frequency diagram for hurricanes

- What does a frequency diagram show? What are the axes?
  - Shows the number of times a variable took a particular value (or group of values)
  - X-axis would be “number of hurricanes in a year”
  - Y-axis would be “count of number of times that number of hurricanes occurred in a year” or “frequency”
Lecture 3 Review - Probability

- Definition of Probability
- Odds
- Conditional Probability
  - Explained with Venn Diagrams
  - Explained with Tables
  - Explained with a Tree
  - AND rule – explained with a tree
  - OR rule – explained with Venn diagrams, then with tree
- Total probability rule
- Bayes’ Rule
- Independent events
Probability

- What is “probability” used to describe?
  - Events that are inherently uncertain (e.g. coin toss, hurricane track)
  - Imperfect knowledge about things that are knowable (e.g. population of U.S., global temperature increase)
  - Subjective uncertainty about the future (e.g. election outcome, Greek default)

- Probability always falls between…
  - Zero and One

- What is the complement of an event, A?
  - The event that A does not occur (A-bar)
  - \( P(A-bar) = 1 - P(A) \)
Odds

- Just another way to write about or describe a probability
- What are the odds of an event, \( W(A) \), in words?
  - The ratio of the probability of an event to the probability of its complement
  - The equation: 
    \[
    W(A) = \frac{P(A)}{P(A)} = \frac{P(A)}{1-P(A)}
    \]

- Two quick sample problems:
  - What are the odds that if you randomly pick a card from a deck, it will be a diamond?
    - \( P(A) = 13/52 = 0.25 \), \( P(A-bar) = 0.75 \); \( W(A) = 0.25/0.75 = "1 to 3 against" \)
  - What are the odds that if you randomly pick a card from a deck, it will be the king of spades?
    - \( P(A) = 1/52 \), \( P(A-bar) = 51/52 \);
    - \( W(A) = (1/52)/(51/52) = "1 to 51 against" \)
Conditional Probability

• What is conditional probability? Why do we use it?
  • Conditional probability is the “probability of A given B” (or probability of A given that B has occurred)
    • Denoted P(A|B)
    • If events are related (correlated) then an event A might be more or less likely if we assume that event B has occurred
      • E.g. Likelihood of getting the flu if your neighbor already has it

• What is a “base rate”?  
  • The probability an event (A) will occur before learning new information
    • P(A)

• What is a “posterior” probability?  
  • The probability of that event (A) occurring given that a related event (B) occurs
    • P(A|B)
Conditional Probability

- Think about this using a Venn Diagram
  - The event that A occurs given B is the area where A and B intersect divided by the full area of B

- $A = \text{Got an A in class}$
- $B = \text{Came to Discussion}$
  - Probability that a student got an A given that they came to lecture is prob. that they came to discussion and got an A, divided by the total number of students that came to lecture
Conditional Probability

• Think about this using a table
• A = Got an A in class
• B = Came to Discussion
  • Probability that a student got an A given that they came to lecture:
  • Probability that they came to discussion and got an A, divided by the total number of students that came to lecture
  
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P(A|B) = P(A and B)/P(B)
```
  • P(got A | attended lecture)
    = P(got an A and came to lecture)/P(came to lecture)
    = (7/110)/(10/110)
    = 7/10
  • Probability that a student got an A, given that they came to lecture was 0.7

| Grade | Attend Disc? | |
|-------|--------------|
| A     | Yes | No | Total |
| A     | 7   | 23 | 30    |
| <=B   | 3   | 77 | 80    |
| Total | 10  | 100| 110   |
Conditional Probability

- What is the probability that a student came to lecture, given that they got an A?
  - \( P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \)
  - \( P(\text{attended lecture} \mid \text{got A}) = \frac{P(\text{got an A and came to lecture})}{P(\text{got an A})} \)
  - \( = \frac{7}{110}/\frac{30}{110} \)
  - \( = \frac{7}{30} \)
  - Probability that a student came to lecture, given that they got an A was 0.23

- **IMPORTANT INSIGHT**: \( P(A|B) \) is not the same as \( P(B|A) \)
Conditional Probability

- Think of this as a tree:
- Probability of A, given B

\[
P(A \text{ and } B) = P(B) \times P(A|B)
\]

- Gives you this equation: \( P(B) \times P(A|B) = P(A \text{ and } B) \)
- Re-arrange to get equation from before
  - \( P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \)
Conditional Probability – AND rule

- Illustrated on previous slide
- \( P(A \text{ and } B) = P(A|B) \cdot P(B) \)
- \( P(A \text{ and } B) = P(A|B) \cdot P(B) = P(B|A) \cdot P(A) \)
  - To understand why this is the same, imagine we had put \( P(A) \) first in the tree diagram on slide 13
  - We would get \( P(A) \cdot P(B|A) = P(A \text{ and } B) \)
Conditional Probability – OR Rule

- How could you calculate the probability of A or B occurring, i.e. $P(A \text{ or } B)$?
  - Hint: Draw a Venn Diagram

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
  - We subtract $P(A \text{ and } B)$ because we don’t want to double-count
  - Using our equation for $P(A \text{ and } B)$, we can also write this as:
    - $P(A \text{ or } B) = P(A) + P(B) - P(B) \cdot P(A|B) = P(A) + P(B) - P(A) \cdot P(B|A)$
Conditional Probability – OR Rule

- What’s another way to compute \( P(A \text{ or } B) \)?
  - Hint: Draw a tree diagram

- The probability that \( A \) or \( B \) happen is one minus the probability that neither of them happen
- \( P(A \text{ or } B) = 1 - P(A\text{-bar} \text{ AND } B\text{-bar}) \)
Conditional Probability Recap

- Conditional probability
  - \( P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \)

- AND rule
  - \( P(A \text{ and } B) = P(B) \cdot P(A|B) = P(A) \cdot P(B|A) \)

- OR rule
  - \( P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \)
  - \( P(A \text{ or } B) = 1 - P(A-bar \text{ and } B-bar) \)

- Tree diagrams help to work through these equations logically
Total Probability Rule

- What is the Total Probability Rule?
  - The probability of A can be considered to have two mutually exclusive components:
    - The probability of A if B has occurred, i.e. \( P(A|B) \cdot P(B) \)
    - The probability of A if B has not occurred, i.e. \( P(A|B\bar{\text{\text{}}}) \cdot P(B\bar{\text{\text{}}}) \)
  - \( P(A) = P(A|B) \cdot P(B) + P(A|B\bar{\text{\text{}}}) \cdot P(B\bar{\text{\text{}}}) \)
Bayes’ Rule

• In what two cases do you want to use Bayes’ Rule?
  • Use this to convert from $P(A|B)$ to $P(B|A)$
  • Use to determine a prior probability, $P(B)$, given information $A$

• Deriving Bayes’ Rule
  • We know $P(A \text{ and } B) = P(A|B)*P(B) = P(B|A)*P(A)$
    • Set these equal and solve for $P(B|A)$
    • $P(B|A) = (P(A|B)*P(B))/P(A)$
    • Now substitute in the total probability equation for $P(A)$

$$
\frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|\overline{B}) \cdot P(\overline{B})}
$$
Conditional Probability Example

• The UMD School of Public Policy is having an event for Masters students and has asked students to RSVP if they can attend. 60% of the students RSVP to the event. If a student has RSVPed, the likelihood he or she will attend is 0.85. If a student has not RSVPed, there is a 10% chance they will show up anyway. How many students attend the event? Given that a person attended, what was the probability that they RSVPed? Assume for this problem that there are 300 Masters students in the program.
Conditional Probability Example

- Solve using a tree diagram

\[
\begin{align*}
P(\text{Attend}|\text{RSVP}) &= 0.85 \\
0.6 \times 0.85 &= 0.51 \text{ (153 students)}
\end{align*}
\]

\[
\begin{align*}
P(\text{Not Attend}|\text{RSVP}) &= 0.15 \\
0.6 \times 0.15 &= 0.09 \text{ (27 students)}
\end{align*}
\]

\[
\begin{align*}
P(\text{Attend}|\text{No RSVP}) &= 0.1 \\
0.4 \times 0.1 &= 0.04 \text{ (12 students)}
\end{align*}
\]

\[
\begin{align*}
P(\text{Not Attend}|\text{No RSVP}) &= 0.9 \\
0.4 \times 0.9 &= 0.36 \text{ (108 students)}
\end{align*}
\]

- 153 + 12 = 165; 165 students attend the event
- 153/165 = 0.927; 92.7% probability that you RSVPed, given that you attended
Conditional Probability Example

- Solve using Bayes’ Rule
  - $P(\text{RSVP})=0.6$
  - $P(\text{NoRSVP})=0.4$
  - $P(\text{Attend}|\text{RSVP})=0.85$
  - $P(\text{NotAttend}|\text{RSVP})=0.15$
  - $P(\text{Attend}|\text{NoRSVP})=0.1$
  - $P(\text{Not Attend}|\text{NoRSVP})=0.9$
  - $P(\text{RSVP}|\text{Attend})=?$

- $P(B|A)=$
  - $P(\text{RSVP}|\text{Attend})$ \[
  \frac{P(\text{Attend}|\text{RSVP}) \cdot P(\text{RSVP})}{P(\text{Attend}|\text{RSVP}) \cdot P(\text{RSVP}) + P(\text{Attend}|\text{NoRSVP}) \cdot P(\text{NoRSVP})}
  \]

- $P(\text{RSVP}|\text{Attend})=$
  - $(P(\text{Attend}|\text{RSVP}) \cdot P(\text{RSVP}))/P(\text{Attend}|\text{RSVP}) \cdot P(\text{RSVP})$
  - $+P(\text{Attend}|\text{NoRSVP}) \cdot P(\text{NoRSVP})$

- $=(0.85 \cdot 0.6)/(0.85 \cdot 0.6 + 0.1 \cdot 0.4)$
- $=0.927$
- 92.7% probability that the person RSVPed, given that they attended
Independence

• What does it mean if event B and event A are independent? Given an example of independent events.
  • Knowing the probability that event B has occurred does not change the probability of event A
  • Knowledge of one is of no value in assessing the probability of the other
  • Formally: \( P(A|B)=P(A) \)
  • Example: Given that I am over five feet tall, what is the likelihood it will rain next Tuesday?
Independence

• Which of these are independent:
  • The probability that a coin lands on heads the first toss and the probability that a coin lands on heads on the second toss
    • Independent
  • The probability that the beach will be crowded and the probability that it will rain
    • Not independent
  • The probability that you will be taller than average, and the probability that your parents are both taller than average
    • Not independent
  • The probability that your first child is a boy and the probability that your second child is a boy
    • Independent
  • The probability that it is raining and the probability that you will understand this material
    • Independent
Independence – AND

• How do we calculate $P(A \text{ and } B)$ for independent events?
  • Hint: You can simplify from the formula given for conditional probability
  • $P(A \text{ and } B) = P(A|B)P(B) = P(A)*P(B)$
  • $P(A \text{ and } B \text{ and } C \text{ and } D) = P(A)*P(B)*P(C)*P(D)$
Independence – OR

• How do we calculate $P(A \text{ or } B)$ for independent events? (Draw a Venn Diagram)
  - $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

  ![Venn Diagram]

  • And subbing in the equation for $P(A \text{ and } B)$
  • $= P(A) + P(B) - P(A | B)P(B)$
  • And simplifying since we know the events are independent
  • $= P(A) + P(B) - P(A) \times P(B)$
Independence – OR

• What is an alternative way to solve for \( P(A \text{ or } B) \) for independent events?
  
  • The only alternative to \( A \) or \( B \) happening would be for NEITHER \( A \) nor \( B \) to happen, so we look at the probability that both do not occur
  
  • \( P(A \text{ or } B) = 1 - P(A\text{-bar}) P(B\text{-bar}) \)

• How do we extend this to multiple events?
  
  • \( P(A \text{ or } B \text{ or } C) = 1 - P(A\text{-bar}) P(B\text{-bar}) P(C\text{-bar}) \)
Independent Probability Example

- My friend owns three bikes, all of which have a 95% probability of making it to work without issue (flat tire, broken chain, etc.). If something breaks, she’s late to work. She bikes to work using a different bike each day, Monday through Wednesday. What is the probability that she is late to work at least one day because of her bike?
Independent Probability Example

- Solve using a Tree Diagram

\[
P(\text{NoIssue}) = 0.95
\]

\[
P(\text{Issue}) = 0.05
\]

\[
P(\text{NoIssue}) = 0.95
\]

\[
P(\text{Issue}) = 0.05
\]

\[
P(\text{NoIssue}) = 0.95
\]

\[
P(\text{Issue}) = 0.05
\]

\[
P(\text{NoIssue}) = 0.95
\]

\[
P(\text{Issue}) = 0.05
\]

\[
P(\text{NoIssue}) = 0.95
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\[
P(\text{Issue}) = 0.05
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\[
P(\text{NoIssue}) = 0.95
\]

\[
P(\text{Issue}) = 0.05
\]

\[
P(\text{NoIssue}) = 0.95
\]

\[
P(\text{Issue}) = 0.05
\]

\[
P(\text{NoIssue}) = 0.95
\]

\[
P(\text{Issue}) = 0.05
\]
Independent Probability Example

• Solve using AND or OR equations

• \( P(A \text{ or } B) \)
  \[ = 1 - P(A-\text{bar and B-\text{bar}}) \]
  \[ = 1 - P(A-\text{bar}) \times P(B-\text{bar}) \]

• \( P(\text{Issue Day 1 or Issue Day 2 or Issue Day 3}) \)
  \[ = 1 - (P(\text{No Issue Day 1 and No Issue Day 2 and No Issue Day 3})) \]
  \[ = 1 - (P(Nol\text{Issue1}) \times P(Nol\text{Issue2}) \times P(Nol\text{Issue3})) \]
  \[ = 1 - 0.95 \times 0.95 \times 0.95 \]
Probability Equation Recap

- **Conditional Probability**
  - \( P(A|B) = \frac{P(A \text{ and } B)}{P(B)} \)
  - **AND rule**
    - \( P(A \text{ and } B) = P(B) \cdot P(A|B) \)
    - \( P(A \text{ and } B) = P(A) \cdot P(B|A) \)
  - **OR rule**
    - \( P(A \text{ or } B) \)
      - \( = P(A) + P(B) - P(A \text{ and } B) \)
      - \( = 1 - P(A-\text{bar} \text{ and } B-\text{bar}) \)
  - **Bayes’ Rule:** \( P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c)} \)

- **Independent Events**
  - \( P(A|B) = P(A) \)
  - **AND rule**
    - \( P(A \text{ and } B) = P(A) \cdot P(B) \)
  - **OR rule**
    - \( P(A \text{ or } B) \)
      - \( = P(A) + P(B) - P(A \text{ and } B) \)
      - \( = P(A) + P(B) - P(A) \cdot P(B) \)
      - \( = 1 - P(A-\text{bar} \text{ and } B-\text{bar}) \)
      - \( = 1 - P(A-\text{bar}) \cdot P(B-\text{bar}) \)
Readings

Chapter 5, pg. 132

- Short definition of independence
- If you flip a penny 10 times and it come up tails all 10 of those times, what is the probability that the next flip will also be tails?
Readings

• Ch14
• Ch15
• Ch16
Thinking Fast and Slow: Ch 14

• Tom W is a graduate student at the main university in your state. Please rank the following nine fields of graduate specialization in order of the likelihood that Tom W is now a student in each of these fields. Use 1 for the most likely, 9 for the least likely.

• business administration
• computer science
• engineering
• humanities and education
• law
• medicine
• library science
• physical and life sciences
• social science and
• social work

Kahneman
Thinking Fast and Slow: Ch 14

• The following is a personality sketch of Tom W written during Tom’s senior year in high school by a psychologist, on the basis of psychological tests of uncertain validity:

• Tom W is of high intelligence, although lacking in true creativity. He has a need for order and clarity, and for neat and tidy systems in which every detail finds its appropriate place. His writing is rather dull and mechanical, occasionally enlivened by somewhat corny puns and flashes of imagination of the sci-fi type. He has a strong drive for competence. He seems to have little feel and little sympathy for other people, and does not enjoy interacting with others. Self-centered, he nonetheless has a deep moral sense.

• Now please take a sheet of paper and rank the nine fields of specialization listed below by how similar the description of Tom W is to the typical graduate student in each of the following fields. Use 1 for the most likely and 9 for the least likely.

Kahneman,
Thinking Fast and Slow: Ch 14

- What do you think happens when people do the second ranking?
  - They change their ranking and put computer science, engineering, etc. closer to the top

- Why might this be unreasonable?
  - The story said that it was of uncertain validity
  - But people use it and ignore the prior probabilities
Thinking Fast and Slow: Ch 14

• How does conditional probability apply here?
  • We could update the probability of his being in each major, given new information about him
  • However, we still can’t ignore prior probability
  • This accounts for the fact that the field might be quite small.
Thinking Fast and Slow: Ch 15

• Linda is thirty-one years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in antinuclear demonstrations.

• Which alternative is more probable?
  • Linda is a bank teller.
  • Linda is a bank teller and is active in the feminist movement.
Thinking Fast and Slow: Ch 16

• A cab was involved in a hit-and-run accident at night. Two cab companies, the Green and the Blue, operate in the city.
  • 85% of the cabs in the city are Green and 15% are Blue.
  • A witness identified the cab as Blue. The court tested the reliability of the witness under the circumstances that existed on the night of the accident and concluded that the witness correctly identified each one of the two colors 80% of the time and failed 20% of the time.
• What would you say is the probability of the cab being blue if there had been no witness?
• What is the probability given the witness testimony?
Thinking Fast and Slow: Ch 16

- Use Bayes’ Rule, \( P(B|A) = \frac{P(A|B) \cdot P(B)}{P(A|B) \cdot P(B) + P(A|\bar{B}) \cdot P(\bar{B})} \)

- \( P(B|A) = \) Probability the cab was blue, given the witness said it was blue (this is what we’re trying to find out)

- \( P(A|B) = \) Probability the witness said it was blue, given that it was blue (80% witness reliability)

- \( P(B) = \) Probability the cab was blue (15% of cabs in the city are blue)

\[
= \frac{(0.8 \times 0.15)}{(0.8 \times 0.15 + 0.2 \times 0.85)}
\]

\[
= 0.41
\]

- There is a 41% chance the cab was blue, given witness testimony
According to the Department of Defense, the U.S. national missile defense system is designed to destroy up to five incoming warheads. It would accomplish this by launching four interceptors against each incoming warhead.

1. Assume that North Korea launches a missile armed with a single warhead at Los Angeles. If each interceptor has a 70 percent kill probability (i.e., chance of destroying an incoming warhead), what is the probability that a single incoming warhead would be destroyed?

2. Now assume that North Korea launches five missiles, each armed with a single warhead, all targeted on Los Angeles. What is the probability that Los Angeles will be destroyed?

3. What key assumption did you make in questions 1.1 and 1.2? Do you think this is a reasonable assumption? Can you think of any circumstances in which this assumption might be invalid?

4. Department of Defense claims that there is a 99 percent chance that all five warheads would be destroyed. What did DoD assume about the kill probability of each interceptor?
Problem Set 3 – Problem 2

A bill introduced in the New York Senate required that any drug screening test must “have a degree of accuracy of at least 95 percent” and “positive test results must then be confirmed by an independent test, using a fundamentally different method and having a degree of accuracy of 98%.” The bill did not define "accuracy", so assume that it refers to both sensitivity and specificity (i.e., the true positive and true negative rates). An anonymous survey in 2010 suggests that about 6.6% of adults over the age of 25 are current users of illegal drugs (including marijuana).

1. In a random sample of 1000 workers over the age of 25, how many, on average, will be drug users?
2. Of the drug users, how many will test positive on the first test?
3. Of the non-users, how many will test positive on the first test?
4. How many workers will test positive? Of these what fraction are drug users? What is the probability that someone is a drug user, given that they tested positive on the first test?
5. Of the drug users, how many will test positive on both the first and second tests?
6. Of the non-users, how many will test positive on both the first and second tests?
7. How many workers will test positive on both tests? Of these, what fraction are drug users? What is the probability that someone is a drug user, if they test positive on both tests?
8. What is the probability that someone is not a drug user, if they test positive on both tests?
9. What is the probability that a drug user will escape detection--that is, that they will test negative on either the first or the second test?
The National Cancer Institute recommends annual mammograms for women beginning at age 40. A study of the accuracy of mammography in diagnosing whether breast tumors are benign or malignant found that, on average, mammograms indicated the presence of a tumor in 79.2% of women who had breast cancer. For women who did not have cancer, 9.6% of mammograms indicated the presence of a tumor.

1. According to NCI, 1 in 68 women will develop breast cancer between the ages of 40 and 49. If a woman in this age range has a positive mammogram, what is the probability that she has breast cancer?

2. It has been discovered that the incidence of breast cancer in women with an altered BRAC1 or BRAC2 gene is ten times higher than in the general population. If a woman aged 40-49 has such a gene, what is the probability that she has breast cancer given a positive mammogram?
Questions?